

# NONLINEAR DEFORMED $\mathfrak{su}(2)$ ALGEBRAS INVOLVING TWO DEFORMING FUNCTIONS \*

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## Abstract

The most common nonlinear deformations of the  $\mathfrak{su}(2)$  Lie algebra, introduced by Polychronakos and Roček, involve a single arbitrary function of  $J_0$  and include the quantum algebra  $\mathfrak{su}_q(2)$  as a special case. In the present contribution, less common nonlinear deformations of  $\mathfrak{su}(2)$ , introduced by Delbecq and Quesne and involving two deforming functions of  $J_0$ , are reviewed. Such algebras include Witten's quadratic deformation of  $\mathfrak{su}(2)$  as a special case. Contrary to the former deformations, for which the spectrum of  $J_0$  is linear as for  $\mathfrak{su}(2)$ , the latter give rise to exponential spectra, a property that has aroused much interest in connection with some physical problems. Another interesting algebra of this type, denoted by  $\mathcal{A}_q^+(1)$ , has two series of  $(N+1)$ -dimensional unitary irreducible representations, where  $N = 0, 1, 2, \dots$ . To allow the coupling of any two such representations, a generalization of the standard Hopf axioms is proposed. The resulting algebraic structure, referred to as a two-colour quasitriangular Hopf algebra, is described.

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# 1 Introduction

Quantized universal enveloping algebras, also called  $q$ -algebras, refer to some specific deformations of (the universal enveloping algebra of) Lie algebras, to which they reduce when the deforming parameter  $q$  (or set of deforming parameters) goes to one [1]. The simplest example of  $q$ -algebra,  $\text{su}_q(2) \equiv U_q(\text{su}(2))$ , was first introduced by Sklyanin, and by Kulish and Reshetikhin [2]. It has found a lot of applications in various branches of physics since its realization in terms of  $q$ -bosonic operators was proposed by Biedenharn and Macfarlane [3].

The  $\text{su}_q(2)$  algebra is a special case of more general deformations of  $\text{su}(2)$ , independently introduced by Polychronakos and Roček [4]. They involve one arbitrary function  $f(J_0)$  in the commutator of  $J_+$  with  $J_-$ , and their representation theory is characterized by a rich variety of phenomena, whose interest in particle physics has been stressed.

More recently, deformations of  $\text{su}(2)$  involving two deforming functions  $F(J_0)$  and  $G(J_0)$  in the commutator of  $J_+$  with  $J_-$  and in that of  $J_0$  with  $J_+$  or  $J_-$ , respectively, have been proposed by Delbecq and Quesne [5, 6, 7]. It is the purpose of the present contribution to review the construction and representation theory of such algebras, and to show how the problem of endowing some of them with a Hopf algebraic structure can be addressed [8].

## 2 Nonlinear deformed $\text{su}(2)$ algebras

Polychronakos-Roček algebras (PRA's) are associative algebras over  $\mathbb{C}$ , generated by three operators  $j_0 = (j_0)^\dagger$ ,  $j_+$ , and  $j_- = (j_+)^\dagger$ , satisfying the commutation relations [4]

$$[j_0, j_+] = j_+, \quad [j_0, j_-] = -j_-, \quad [j_+, j_-] = f(j_0), \quad (1)$$

where  $f(z)$  is a real, parameter-dependent function of  $z$ , holomorphic in the neighbourhood of zero, and going to  $2z$  for some values of the parameters. These algebras have a Casimir operator given by  $c = j_- j_+ + h(j_0) = j_+ j_- + h(j_0) - f(j_0)$ , in terms of another real function  $h(z)$ , related to  $f(z)$  through the equation  $h(z) - h(z-1) = f(z)$ . An explicit expression for  $h(z)$  has been given by Delbecq and Quesne [5] in terms of Bernoulli polynomials and Bernoulli numbers.

For all PRA's, the spectrum of  $j_0$  is linear as in the special case of the  $q$ -algebra  $\text{su}_q(2)$ . The latter corresponds to  $f(j_0) = [2j_0]_q$ , and  $h(j_0) = [j_0]_q[j_0 + 1]_q$ , with  $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$ , and  $q \in \mathbb{R}^+$  (the case where  $q$  is a phase will not be considered here, as throughout the present work we shall restrict the parameters to real values) [3].

Delbecq-Quesne algebras (DQA's) differ from PRA's by the replacement of (1) by [5]

$$[J_0, J_+] = G(J_0)J_+, \quad [J_0, J_-] = -J_-G(J_0), \quad [J_+, J_-] = F(J_0), \quad (2)$$

where  $J_0 = (J_0)^\dagger$ ,  $J_- = (J_+)^\dagger$ , and the commutators involve two real, parameter-dependent functions of  $z$ ,  $F(z)$  and  $G(z)$ , holomorphic in the neighbourhood of zero, and going to  $2z$  and 1 for some values of the parameters, respectively. These functions are further restricted by the assumption that the algebras have a Casimir operator given by  $C = J_- J_+ + H(J_0) = J_+ J_- + H(J_0) - F(J_0)$ , in terms of some real function  $H(z)$ , holomorphic in the neighbourhood of zero. The latter restriction implies that  $F(z)$ ,  $G(z)$ , and  $H(z)$  satisfy the consistency condition  $H(z) - H(z - G(z)) = F(z)$ .

Since for  $G(J_0) = 1$ , DQA's reduce to PRA's, the first significant case corresponds to  $G(J_0) = 1 + (1 - q)J_0$ , where  $q \in \mathbb{R}^+$ . In such a case, it has been shown that there exist  $(\lambda - 1)$ -parameter algebras  $\mathcal{A}_{\alpha_2 \alpha_3 \dots \alpha_{\lambda-1} q}^+(\lambda, 1)$ , for which the functions  $F(J_0)$  and  $H(J_0)$  are polynomials of degree  $\lambda$  in  $J_0$  [5]. In particular, for  $\lambda = 2$  and 3, one finds the algebras  $\mathcal{A}_q^+(2, 1)$  [6] and  $\mathcal{A}_{p,q}^+(3, 1)$  [7], for which  $F(J_0) = 2J_0(1 + (1 - q)J_0)$ ,  $H(J_0) = 2(1 + q)^{-1}J_0(J_0 + 1)$ , and  $F(J_0) = 2J_0(1 + (1 - q)J_0)(1 - (1 - p)J_0)$ ,  $H(J_0) = 2((1 + q)(1 + q + q^2))^{-1}J_0(J_0 + 1)(1 + (p + q)q - (1 - p)(1 + q)J_0)$ , respectively. The former is equivalent to Witten's first deformation of  $\text{su}(2)$  [9].

The representation theory of the DQA's can be dealt with as that of  $\text{su}_q(2)$ , or more generally of the PRA's. Considering the case where  $G(J_0) = 1 + (1 - q)J_0$ , and denoting by  $|cm\rangle$  a simultaneous eigenvector of the commuting Hermitian operators  $C$  and  $J_0$ , associated with the eigenvalues  $c$  and  $m$  respectively, it can be proved [5] that  $J_+^n |cm\rangle$  (resp.  $J_-^n |cm\rangle$ ),  $n \in \mathbb{N}^+$ , is either the null vector or a simultaneous eigenvector of  $C$  and  $J_0$ , corresponding to the eigenvalues  $c$  and  $m q^{-n} - (1 - q^{-n})/(1 - q)$  (resp.  $m q^n - (1 - q^n)/(1 - q)$ ). Hence, the spectrum of  $J_0$  is exponential, instead of linear as for the PRA's.

Moreover, if the starting  $m$  value belongs to the interval  $((q - 1)^{-1}, +\infty)$  (resp.  $(-\infty, (q - 1)^{-1})$ ), then all the  $J_0$  eigenvalues obtained by successive applications of  $J_+$  or  $J_-$  upon  $|cm\rangle \neq 0$  belong to the same interval and  $J_+$  (resp.  $J_-$ ) is a raising generator, whereas if  $m = (q - 1)^{-1}$ , then neither  $J_+$  nor  $J_-$  change the  $J_0$  eigenvalue. The unirreps therefore separate into two classes according to whether the eigenvalues of  $J_0$  are contained in the interval  $(-\infty, (q - 1)^{-1})$ , or in the interval  $((q - 1)^{-1}, +\infty)$ .

In general, they may fall into one out of four categories: (i) infinite-dimensional unirreps with a lower bound  $-j$ , (ii) infinite-dimensional unirreps with an upper bound  $J$ , (iii) infinite-dimensional unirreps with neither lower nor upper bounds, and (iv) finite-dimensional unirreps with both lower and upper bounds,  $-j$  and  $J$  (where in general  $j \neq J$ ). In addition, there is a trivial one-dimensional unirrep corresponding to  $m = (q - 1)^{-1}$ .

The exponential character of the  $J_0$  spectrum in DQA representations may be of interest in various physical problems, wherein such spectra have been encountered, such as alternative Hamiltonian quantizations, exactly solvable potentials,  $q$ -deformed supersymmetric quantum mechanics, and  $q$ -deformed interacting boson models [10].

### 3 The algebra $\mathcal{A}_q^+(1)$

Another example of DQA, for which the function  $G(J_0)$  is linear, has been recently constructed [8]. Contrary to those considered in the previous section, this algebra, denoted by  $\mathcal{A}_q^+(1)$ , is defined in terms of functions  $F(J_0)$  and  $H(J_0)$  that are not polynomials, but infinite series in  $J_0$ ,

$$\begin{aligned} F(J_0) &= -\frac{(G(J_0))^2 - (G(J_0))^{-2}}{q - q^{-1}}, \\ H(J_0) &= \frac{q^{-1}(G(J_0))^2 + q(G(J_0))^{-2} - q - q^{-1}}{(q - q^{-1})^2}, \end{aligned} \quad (3)$$

with  $G(J_0) = 1 + (1 - q)J_0$ . Since the transformation  $q \rightarrow q^{-1}$ ,  $J_0 \rightarrow -qJ_0$ ,  $J_{\pm} \rightarrow J_{\mp}$  is an automorphism of  $\mathcal{A}_q^+(1)$ , the parameter values may be restricted to the range  $0 < q < 1$ .

$\mathcal{A}_q^+(1)$  can be obtained from  $\text{su}_q(2)$ , a special case of PRA, by using a two-valued map  $P_{\delta} : \text{su}_q(2) \rightarrow \mathcal{A}_q^+(1)$ ,  $\delta = \pm 1$ , defined by

$$J_0 = p_{\delta}(j_0), \quad J_+ = j_+, \quad J_- = j_-, \quad \text{where} \quad p_{\delta}(z) \equiv \frac{1 - \delta q^{-z}}{q - 1}. \quad (4)$$

Such a generator map is well defined: it can indeed be easily checked that if  $j_0, j_+, j_-$  satisfy the  $\text{su}_q(2)$  commutation relations, then  $J_0, J_+, J_-$ , given in (4), fulfil those of  $\mathcal{A}_q^+(1)$ . The functions  $p_{\delta}(z)$ ,  $\delta = \pm 1$ , defined in (4), are entire and invertible functions, with  $g(z) = p_{\delta}^{-1}(z) = \ln(G^2(z)) / \ln(q^{-2})$ . If  $z \in \mathbb{R}$ , the range of  $p_{\delta}$  (and consequently the domain of  $p_{\delta}^{-1}$ ) is the interval  $(-\infty, (q - 1)^{-1})$  or  $((q - 1)^{-1}, +\infty)$  according to whether  $\delta = -1$  or  $\delta = +1$ . The function  $g(z)$  is well-behaved everywhere on  $\mathbb{R}$ , except in the neighbourhood of the point  $z = (q - 1)^{-1}$ .

It should be stressed that the use of  $P_{\delta}$ ,  $\delta = \pm 1$ , implies an extension of the well-known deforming functional technique [11] for two reasons: first because here a map between two deformed algebras,  $\text{su}_q(2)$  and  $\mathcal{A}_q^+(1)$ , is considered instead of a map between a Lie algebra and a deformed one, as in the original method; and second because use is made of a two-valued functional, whose inverse is singular, instead of a single-valued one.

It can be easily shown [8] that  $\mathcal{A}_q^+(1)$  has no infinite-dimensional unirrep, but has, for any  $N = 0, 1, 2, \dots$ , two  $(N + 1)$ -dimensional unirreps, which may be distinguished by  $\delta = \pm 1$ . The corresponding spectrum of  $J_0$  is given by  $m^{\delta} = (1 - \delta q^{-(N-2n)/2}) / (q - 1)$ ,  $n = 0, 1, \dots, N$ , with maximum and minimum eigenvalues  $J^{\delta} = (1 - \delta q^{-\delta N/2}) / (q - 1)$ , and  $-j^{\delta} = (1 - \delta q^{\delta N/2}) / (q - 1)$  respectively. The unirrep specified by  $J^+$  (resp.  $J^-$ ) is entirely contained in the interval  $((q - 1)^{-1}, +\infty)$  (resp.  $(-\infty, (q - 1)^{-1})$ ). For both unirreps, the eigenvalue of the Casimir operator is given by  $\langle C \rangle = H(\gamma^{\delta})$ , where  $\gamma^{\delta} = (1 - \delta q^{-N/2}) / (q - 1)$ .

In the carrier space  $V^{J^{\delta}}$  of the unirrep characterized by  $J^{\delta}$ , whose basis vectors are specified by the values of  $J^{\delta}$  and  $m^{\delta}$ , the  $\mathcal{A}_q^+(1)$  generators are represented by

some linear operators  $\Phi^{J^\delta}(A)$ ,  $A \in \mathcal{A}_q^+(1)$ , defined by

$$\begin{aligned}
\Phi^{J^\delta}(J_0) |J^\delta, m^\delta\rangle &= m^\delta |J^\delta, m^\delta\rangle = \left(\frac{N}{2} - n\right) |J^\delta, m^\delta\rangle, \\
\Phi^{J^\delta}(J_-) |J^\delta, m^\delta\rangle &= \sqrt{H(\gamma^\delta) - H(qm^\delta - 1)} |J^\delta, qm^\delta - 1\rangle \\
&= \sqrt{[n+1]_q [N-n]_q} |J^\delta, qm^\delta - 1\rangle, \\
\Phi^{J^\delta}(J_+) |J^\delta, m^\delta\rangle &= \sqrt{H(\gamma^\delta) - H(m^\delta)} |J^\delta, q^{-1}(m^\delta + 1)\rangle \\
&= \sqrt{[n]_q [N-n+1]_q} |J^\delta, q^{-1}(m^\delta + 1)\rangle. \tag{5}
\end{aligned}$$

The generator mapping  $P_\delta$  can be used to transfer the quasitriangular Hopf structure of  $\text{su}_q(2)$  to  $\mathcal{A}_q^+(1)$  [8]. One gets in this way a double quasitriangular Hopf structure, with comultiplication, counit, antipode maps, and universal  $\mathcal{R}$ -matrix given by

$$\begin{aligned}
\Delta_\delta(J_0) &= (q-1)^{-1} (1 \otimes 1 - \delta G(J_0) \otimes G(J_0)), \\
\Delta_\delta(J_\pm) &= \delta \left( J_\pm \otimes (G(J_0))^{-1} + G(J_0) \otimes J_\pm \right), \\
\epsilon_\delta(J_0) &= (1-\delta)(q-1)^{-1}, \quad \epsilon_\delta(J_\pm) = 0, \\
S_\delta(J_0) &= -J_0(G(J_0))^{-1}, \quad S_\delta(J_+) = -qJ_+, \quad S_\delta(J_-) = -q^{-1}J_-, \\
\mathcal{R}^\delta &= q^{2 \log_q(\delta G(J_0)) \otimes \log_q(\delta G(J_0))} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{n(n-1)/2} \\
&\quad \times ((G(J_0))^{-1} J_+ \otimes G(J_0) J_-)^n, \tag{6}
\end{aligned}$$

respectively. Both  $(\Delta_+, \epsilon_+, S_+, \mathcal{R}^+)$ , and  $(\Delta_-, \epsilon_-, S_-, \mathcal{R}^-)$  satisfy the Hopf and quasitriangularity axioms, but the former are only valid for the representations of  $\mathcal{A}_q^+(1)$  with eigenvalues of  $J_0$  in the interval  $((q-1)^{-1}, +\infty)$ , whereas the latter act in  $(-\infty, (q-1)^{-1})$ .

## 4 Two-colour quasitriangular Hopf structure of $\mathcal{A}_q^+(1)$

The double Hopf structure considered in the previous section allows one to couple any two  $\mathcal{A}_q^+(1)$  unirreps characterized by  $J_1^+$  and  $J_2^+$  (resp.  $J_1^-$  and  $J_2^-$ ), and with respective carrier spaces  $V^{J_1^+}$  and  $V^{J_2^+}$  (resp.  $V^{J_1^-}$  and  $V^{J_2^-}$ ), to obtain a reducible representation of the same in  $V^{J_1^+} \otimes V^{J_2^+}$  (resp.  $V^{J_1^-} \otimes V^{J_2^-}$ ). No coupling of two unirreps of the types  $J_1^+$  and  $J_2^-$ , or  $J_1^-$  and  $J_2^+$ , is however possible.

To allow such types of couplings, it is necessary to extend the double Hopf structure of  $\mathcal{A}_q^+(1)$  [8]. This can be accomplished by considering the ‘transmutation’ operators  $T^{J^\delta} : V^{J^\delta} \rightarrow V^{J^{-\delta}}$ , which change the basis states of an  $(N+1)$ -dimensional

unirrep, characterized by  $J^\delta$ , into those of its partner with the same dimension, specified by  $J^{-\delta}$ , i.e.,  $T^{J^\delta} |J^\delta, m^\delta\rangle = |J^{-\delta}, m^{-\delta}\rangle$ . By applying  $T^{J^\delta}$  on both sides of (5), we obtain that for any  $\mathcal{A}_q^+(1)$  generator  $A$ ,  $T^{J^\delta} \Phi^{J^\delta}(A) T^{J^{-\delta}} = \Phi^{J^{-\delta}}(\sigma(A))$ , where  $\sigma : \mathcal{A}_q^+(1) \rightarrow \mathcal{A}_q^+(1)$ , defined by  $\sigma(J_0) = 2(q-1)^{-1} - J_0$ , and  $\sigma(J_\pm) = J_\pm$ , is an involutive automorphism of the algebra  $\mathcal{A}_q^+(1)$ . This clearly shows that at the algebra level, the operator  $\sigma$  is responsible for the transmutation. Defining now  $\sigma_\delta : \mathcal{A}_q^+(1) \rightarrow \mathcal{A}_q^+(1)$ ,  $\delta = \pm 1$ , as  $\sigma_+ = \text{id}$ , and  $\sigma_- = \sigma$ , we note that the basic mapping  $P_\delta$ , defined in (4), satisfies the relation  $\sigma_{\zeta\eta} \circ P_\eta = P_\zeta$ , where  $\zeta, \eta = \pm 1$ .

The comultiplication and antipode maps, as well as the double  $\mathcal{R}$ -matrix of equation (6) can be extended by setting

$$\begin{aligned} \Delta_\delta^{\zeta,\eta}(A) &= (\sigma_{\zeta\delta} \otimes \sigma_{\eta\delta}) \circ \Delta_\delta(A), & S_\delta^\zeta(A) &= \sigma_{\zeta\delta} \circ S_\delta(A), \\ \mathcal{R}^{\zeta,\eta} &= (\sigma_{\zeta\delta} \otimes \sigma_{\eta\delta}) (\mathcal{R}^\delta), \end{aligned} \quad (7)$$

where  $\zeta, \eta, \delta = \pm 1$ , while the counit map  $\epsilon_\delta$ , defined in the same equation, is left unchanged. The results can be written as

$$\begin{aligned} \Delta_\delta^{\zeta,\eta}(J_0) &= (q-1)^{-1} (1 \otimes 1 - \delta\zeta\eta G(J_0) \otimes G(J_0)), \\ \Delta_\delta^{\zeta,\eta}(J_\pm) &= \eta J_\pm \otimes (G(J_0))^{-1} + \zeta G(J_0) \otimes J_\pm, \\ S_\delta^\zeta(J_0) &= (q-1)^{-1} (1 - \zeta\delta (G(J_0))^{-1}), \\ S_\delta^\zeta(J_\pm) &= -q^{\pm 1} J_\pm, \\ \mathcal{R}^{\zeta,\eta} &= q^{2 \log_q(\zeta G(J_0)) \otimes \log_q(\eta G(J_0))} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{n(n-1)/2} ((\zeta G(J_0))^{-1} J_+ \otimes \eta G(J_0) J_-)^n. \end{aligned} \quad (8)$$

It can be easily shown [8] that the generalized comultiplication, counit, and antipode maps,  $\Delta_\delta^{\zeta,\eta}$ ,  $\epsilon_\delta$ ,  $S_\delta^\zeta$ , defined in (6) and (8), transform under  $\sigma_\delta$  as

$$(\sigma_{\mu\zeta} \otimes \sigma_{\nu\eta}) \circ \Delta_\delta^{\zeta,\eta} = \Delta_\delta^{\mu,\nu} \circ \sigma_{\rho\delta}, \quad \epsilon_\delta \circ \sigma_{\delta\zeta} = \epsilon_\zeta, \quad \sigma_{\zeta\eta} \circ S_\delta^\eta = S_\delta^\zeta \circ \sigma_{\mu\delta}, \quad (9)$$

and satisfy the following generalized coassociativity, counit, and antipode axioms,

$$\begin{aligned} (\Delta_\mu^{\zeta,\eta} \otimes \text{id}) \circ \Delta_\delta^{\mu,\nu}(A) &= (\text{id} \otimes \Delta_\rho^{\eta,\nu}) \circ \Delta_\delta^{\zeta,\rho}(A), \\ (\epsilon_\zeta \otimes \sigma_{\eta\delta}) \circ \Delta_\delta^{\zeta,\eta}(A) &= (\sigma_{\zeta\delta} \otimes \epsilon_\eta) \circ \Delta_\delta^{\zeta,\eta}(A) = A, \\ m \circ (S_\zeta^\mu \otimes \sigma_{\mu\eta}) \circ \Delta_\delta^{\zeta,\eta}(A) &= m \circ (\sigma_{\mu\zeta} \otimes S_\eta^\mu) \circ \Delta_\delta^{\zeta,\eta}(A) \\ &= \iota \circ \epsilon_\delta(A), \end{aligned} \quad (10)$$

where  $A$  denotes any element of  $\mathcal{A}_q^+(1)$ ,  $m$  and  $\iota$  are the multiplication and unit maps of  $\mathcal{A}_q^+(1)$ ,  $\delta, \zeta, \eta, \mu, \nu, \rho$  take any values in the set  $\{-1, +1\}$ , and no sum-

mation over repeated indices is implied. Moreover,  $\Delta_\delta^{\zeta,\eta}$  and  $\epsilon_\delta$  are algebra homomorphisms, while  $S_\delta^\zeta$  is both an algebra and a coalgebra antihomomorphism.

By using the generalized coproduct  $\Delta_\delta^{\zeta,\eta}$ , it is now possible to couple any  $(N_1 + 1)$ - and  $(N_2 + 1)$ -dimensional unirreps of  $\mathcal{A}_q^+(1)$ , specified by  $J_1^\zeta$  and  $J_2^\eta$  respectively, to provide two reducible representations in  $V^{J_1^\zeta} \otimes V^{J_2^\eta}$ , which are characterized by  $\delta = +1$  and  $\delta = -1$ , respectively. They can be decomposed into a direct sum of  $(N + 1)$ -dimensional unirreps, specified by  $J^\delta$ , by using some Wigner coefficients  $\langle J_1^\zeta m_1^\zeta, J_2^\eta m_2^\eta | J^\delta m^\delta \rangle_{DQ}$ , given in terms of  $\text{su}_q(2)$  Wigner coefficients by the relation

$$\langle J_1^\zeta m_1^\zeta, J_2^\eta m_2^\eta | J^\delta m^\delta \rangle_{DQ} = \langle \frac{N_1}{2} \frac{N_1}{2} - n_1, \frac{N_2}{2} \frac{N_2}{2} - n_2 | \frac{N}{2} \frac{N}{2} - n \rangle_q. \quad (11)$$

The carrier space of the unirrep  $J^\delta$  in  $V^{J_1^\zeta} \otimes V^{J_2^\eta}$  is therefore spanned by the states

$$|J_1^\zeta J_2^\eta J^\delta m^\delta\rangle = \sum_{m_1^\zeta, m_2^\eta} \langle J_1^\zeta m_1^\zeta, J_2^\eta m_2^\eta | J^\delta m^\delta \rangle_{DQ} |J_1^\zeta, m_1^\zeta\rangle \otimes |J_2^\eta, m_2^\eta\rangle. \quad (12)$$

Turning now to the generalized universal  $\mathcal{R}$ -matrix defined in (7) or (8), it can be easily shown [8] that its four pieces  $\mathcal{R}^{\zeta,\eta}$ ,  $\zeta, \eta = \pm 1$ , are invertible and satisfy the properties

$$\begin{aligned} (\sigma_{\mu\zeta} \otimes \sigma_{\nu\eta}) \left( \mathcal{R}^{\zeta,\eta} \right) &= \mathcal{R}^{\mu,\nu}, \quad \tau \circ \Delta_\delta^{\eta,\zeta}(A) = \mathcal{R}^{\zeta,\eta} \Delta_\delta^{\zeta,\eta}(A) \left( \mathcal{R}^{\zeta,\eta} \right)^{-1}, \\ (\Delta_\zeta^{\lambda,\mu} \otimes \sigma_{\nu\eta}) \left( \mathcal{R}^{\zeta,\eta} \right) &= \mathcal{R}_{13}^{\lambda,\nu} \mathcal{R}_{23}^{\mu,\nu}, \quad (\sigma_{\lambda\zeta} \otimes \Delta_\eta^{\mu,\nu}) \left( \mathcal{R}^{\zeta,\eta} \right) = \mathcal{R}_{13}^{\lambda,\nu} \mathcal{R}_{12}^{\lambda,\mu}, \end{aligned} \quad (13)$$

for any  $A \in \mathcal{A}_q^+(1)$ . From these results, or more simply from the corresponding properties fulfilled by  $\mathcal{R}^\delta$ , one obtains the relations

$$\begin{aligned} \mathcal{R}_{12}^{\zeta,\eta} \mathcal{R}_{13}^{\zeta,\mu} \mathcal{R}_{23}^{\eta,\mu} &= \mathcal{R}_{23}^{\eta,\mu} \mathcal{R}_{13}^{\zeta,\mu} \mathcal{R}_{12}^{\zeta,\eta}, \\ (\epsilon_\zeta \otimes \text{id}) \left( \mathcal{R}^{\zeta,\eta} \right) &= (\text{id} \otimes \epsilon_\eta) \left( \mathcal{R}^{\zeta,\eta} \right) = 1, \\ (S_\zeta^\lambda \otimes \sigma_{\mu\eta}) \left( \mathcal{R}^{\zeta,\eta} \right) &= (\sigma_{\lambda\zeta} \otimes (S_\eta^\mu)^{-1}) \left( \mathcal{R}^{\zeta,\eta} \right) = \left( \mathcal{R}^{\lambda,\mu} \right)^{-1}. \end{aligned} \quad (14)$$

The first relation in (14) shows that the generalized universal  $\mathcal{R}$ -matrix is a solution of the coloured YBE [12], where the ‘colour’ parameters  $\zeta, \eta, \mu$  take discrete values in the set  $\{-1, +1\}$ . We may therefore call  $(\mathcal{A}_q^+(1), +, m, \iota, \Delta_\delta^{\zeta,\eta}, \epsilon_\delta, S_\delta^\zeta, \mathcal{R}^{\zeta,\eta}; \mathbb{C})$  a two-colour quasitriangular Hopf algebra over  $\mathbb{C}$  [8]. As will be shown elsewhere [13], this type of algebraic structure admits generalizations, which will be referred to as coloured quasitriangular Hopf algebras.

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